Nonlinear evolution-type equations and their exact solutions using inverse variational methods

2005 J. Phys. A: Math. Gen. 38 4629

(http://iopscience.iop.org/0305-4470/38/21/008)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 4629–4636 doi:10.1088/0305-4470/38/21/008

Nonlinear evolution-type equations and their exact solutions using inverse variational methods

A H Kara[1] **and C M Khalique**[2]

[1] School of Mathematics and Centre for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa
[2] Department of Mathematical Sciences, Northwest University, Mmabatho, South Africa

E-mail: kara@maths.wits.ac.za and khaliquecm@uniwest.ac.za

Received 11 October 2004, in final form 24 March 2005
Published 10 May 2005
Online at stacks.iop.org/JPhysA/38/4629

Abstract

We present the role of invariants in obtaining exact solutions of differential equations. Firstly, conserved vectors of a partial differential equation (p.d.e.) allow us to obtain reduced forms of the p.d.e. for which some of the Lie point symmetries (in vector field form) are easily concluded and, therefore, provide a mechanism for further reduction. Secondly, invariants of reduced forms of a p.d.e. are obtainable from a variational principle even though the p.d.e. itself does not admit a Lagrangian. In this latter case, the reductions carry all the usual advantages regarding Noether symmetries and double reductions. The examples we consider are nonlinear evolution-type equations such as the Korteweg–deVries equation, but a detailed analysis is made on the Fisher equation (which describes reaction–diffusion waves in biology, *inter alia*). Other diffusion-type equations lend themselves well to the method we describe (e.g., the Fitzhugh Nagumo equation, which is briefly discussed). Some aspects of Painlevé properties are also suggested.

PACS numbers: 02.30.Jr, 02.30.Ik
Mathematics Subject Classification: 37K05, 35Q53, 35A15, 35A25

1. Introduction

The method of 'invariants' to analyse differential equations (d.e.s) is now commonly used; in most cases the invariants referred to are a consequence of conserved forms of the d.e. These provide a way of reducing the d.e. which in the case of partial d.e.s (p.d.e.s) may mean a decrease in the number of independent variables or, as in the case of ordinary d.e.s (o.d.e.s), refers to the order of the d.e. (e.g., see [9] or [3]). In [1, 2], the author uses the invariant idea in conjunction with specific algorithms to anlayse equations, *inter alia*, the KdV, Nagumo and Fisher equations. All of these are evolution-type equations which do not admit Lagrangians, which is somewhat unfortunate as the existence and knowledge of a Lagrangian makes the task of finding invariants easier via Noether's theorem.

Nevertheless, we show that some of these and other equations (such as Burger's equation which models shock wave phenomena) may still be analysed using the Lagrangian method. Furthermore, we explore the idea that a Lie point symmetry generator of a p.d.e. associated with a conserved vector may easily provide a Lie point symmetry of the reduced system so that double reduction of the original p.d.e may be achieved.

We consider some and other of the examples mentioned above which arise in biology, (Fisher's) or in the modelling of shallow water behaviour (KdV).

We only recall some of the more salient features. Consider an rth-order system of p.d.e.s of n independent and m dependent variables, namely,

$$E^{\beta}(x, u, u_{(1)}, \dots, u_{(r)}) = 0, \qquad \beta = 1, \dots, \tilde{m}. \tag{1.1}$$

A conservation law of (1.1) is the equation

$$D_i T^i = 0, \tag{1.2}$$

on the solutions of (1.1). Here the *total differentiation operator* is

$$D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha} \frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha} \frac{\partial}{\partial u_j^{\alpha}} + \cdots, \qquad i = 1, \dots, n. \tag{1.3}$$

The tuple $T = (T^1, \dots, T^n)$ is called a *conserved vector* of (1.1).

Suppose $\mathcal{A}$ is the universal space of differential functions. A Lie–Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \zeta_i^{\alpha} \frac{\partial}{\partial u_i^{\alpha}} + \zeta_{i_1 i_2}^{\alpha} \frac{\partial}{\partial u_{i_1 i_2}^{\alpha}} + \cdots, \tag{1.4}$$

where $\xi^i, \eta^{\alpha} \in \mathcal{A}$ and the additional coefficients are

$$\begin{aligned} \zeta_i^{\alpha} &= D_i(W^{\alpha}) + \xi^j u_{ij}^{\alpha}, \\ \zeta_{i_1 i_2}^{\alpha} &= D_{i_1} D_{i_2}(W^{\alpha}) + \xi^j u_{j i_1 i_2}^{\alpha}, \\ &\vdots \end{aligned} \tag{1.5}$$

and W^{α} is the Lie characteristic function defined by

$$W^{\alpha} = \eta^{\alpha} - \xi^j u_j^{\alpha}. \tag{1.6}$$

In this paper, we will assume that X is a Lie point operator, i.e., ξ and η are functions of x and u and are independent of derivatives of u.

Theorem 1 [7]. *Suppose that X is a Lie–Bäcklund symmetry of the system (1.1) such that the conserved vector* $T = (T^1, \dots, T^n)$ *is invariant under X. Then*

$$X(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i) = 0, \qquad i = 1, \dots, n. \tag{1.7}$$

Definition. *When (1.7) is satisfied, we say that X is associated with T.*

Whilst the notion of association of symmetries and conservation laws and its application in the reduction of o.d.e.s is known, for p.d.e.s, the concept is relatively new. Moreover, its application in the reduction of the p.d.e. is demonstrated here (section 2) for the first time.

A conservation law (1.2) provides a *potential system* corresponding to (1.1) with the introduction of a potential variable (dependent) v, namely, for two independent variables $x_1 = t$ and $x_2 = x$, the potential system is

$$v_x = T^1, \qquad v_t = -T^2. \tag{1.8}$$

2. Symmetries, invariants and double reduction

We present here a range of examples that show that the association of a symmetry X of a system with a conserved vector T provides a 'reduction' of the system which, in turn, has a symmetry being an extension of X (X^v) to include the potential variable. Indeed, X^v can be used to reduce the already reduced system. In the next section, we look at a 'Lagrangian' picture of the reduced forms of the equations even though the equation concerned may be of evolution type.

Example 1: Burger's equation

We consider the double reduction of the well-known equation

$$u_t = u_{xx} + uu_x. \tag{2.1}$$

(a) The symmetry generator corresponding to 'boost', namely, $X = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}$, is associated with $T^1 = u + x/t$, $T^2 = -u_x - \frac{1}{2}u^2 + x^2/(2t^2)$ in the sense of the theorem and definition above (done in [7]) and it can be shown that $X^v = X + k\frac{\partial}{\partial v}$ (k is a constant) is a symmetry of the corresponding potential (and reduced) system

$$v_x = u + x/t, \qquad v_t = u_x + \tfrac{1}{2}u^2 - x^2/(2t^2). \tag{2.2}$$

With $k = 1$, invariants of X^v are $y = t$, $\alpha = u + x/t$ and $\beta = v - x/t$ ($\alpha(y)$ and $\beta(y)$ so that $u_x = -1/t$, $v_x = 1/t$, $u_t = \alpha' + x/t^2$ and $v_t = \beta' - x/t^2$. The u solution of (2.2), then, is $u = (1 - x)/t$.

This solution may be achieved, without much of a difference in the calculations, in the direct reduction of (2.1). However, from what follows below, the usefulness of the 'double reduction' for (2.1) is clear from the invariant solution of (2.1) obtained form the dilation symmetry $G = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}$.

(b) It can be shown that $G^v = G$ above is a Lie point symmetry generator of the system

$$v_x = u + 2x/t, \qquad v_t = u_x + \tfrac{1}{2}u^2 - x^2/(t^2) \tag{2.3}$$

obtained by noting that G is associated with the conserved vector $\left(u + 2x/t, -u_x - \frac{1}{2}u^2 + x^2/t^2\right)$. Thus, a reduction of (2.3) is achieved by the invariants of G^v, namely, $y = x^2/t$, $\alpha = ux$ and $\beta = v$. The system then becomes

$$2y\beta' = \alpha + 2y, \qquad -y\beta' = 2\alpha' - \alpha/y + \tfrac{1}{2}\alpha^2\big/y. \tag{2.4}$$

We, thus, get the first-order o.d.e

$$2\alpha' + \frac{y-2}{2y}\alpha = -\frac{1}{2}\frac{\alpha^2}{y} \tag{2.5}$$

which gives a solution

$$\alpha(y) = \frac{2\sqrt{y}\,\mathrm{e}^{-y/4}}{2k + \sqrt{\pi}\,Erf(\sqrt{y}/2)} \tag{2.6}$$

so that the G invariant solution (self-similar) is obtainable from $u = x\alpha\left(\frac{x^2}{t}\right)$.

If one does not appeal to the association with conservation laws described here, this G invariant solution would require the usual reduction that is described in the note below in which the corresponding second-order o.d.e cannot be analytically solved by any standard method. Also, one may resort to a transformation to the heat equation via the Hopf–Cole transformation. Our method for obtaining this solution avoids both these intricacies.

Notes

(1) A direct reduction of Burger's equation using G would yield the nonlinear second-order o.d.e

$$y^2(4w'' + w') + 2yw'(w - 1) + 2w(1 - w) = 0,$$

where $w = w(y)$ ($y = x^2/t$ and $w = xu$).

(2) The well-known travelling wave solution of (2.2) is obtainable from the generators $\partial/\partial t$ and $\partial/\partial x$ by letting $y = x - ct$ (c is the wave speed) and $w = u$ ($w = w(y)$). With this choice of variables, Burger's equation becomes the second-order o.d.e

$$w'' + ww' + cw' = 0$$

and one integration yields

$$w' + \tfrac{1}{2}w^2 - cw = k, \tag{2.7}$$

where k is a constant. Alternatively, we show that the first-order o.d.e. (2.7) is directly obtainable using the association property above. The generator $Y^v = c\partial/\partial x + \partial/\partial t + \partial/\partial v$ is a Lie point symmetry of the potential system

$$v_x = u, \quad v_t = u_x + \tfrac{1}{2}u^2 \tag{2.8}$$

($c\partial/\partial x + \partial/\partial t$ is associated with the conserved vector $(u, -u_x - \frac{1}{2}u^2)$). The invariants of G^v, namely, $y = x - ct, \alpha = u$ and $\beta = v - t$ reduce (2.8) to the system

$$\beta' = \alpha, \tag{2.9a}$$

$$\alpha' + \tfrac{1}{2}\alpha^2 + c\alpha = 1. \tag{2.9b}$$

Compare (2.7) with (2.9b).

Example 2: Korteweg–deVries equation

A model of shallow behaviour is given by the well-known equation

$$u_t = uu_x + u_{xxx}. \tag{2.10}$$

A conserved vector is $\left(u, -u_{xx} - \frac{1}{2}u^2\right)$ (associated with the symmetry generator $G = c\partial/\partial x + \partial/\partial t$ (c is the wave speed) so that a potential form of (2.10) is

$$v_x = u, \qquad v_t = u_{xx} + \tfrac{1}{2}u^2 \tag{2.11}$$

with a symmetry generator $G^v = c\partial/\partial x + \partial/\partial t + \partial/\partial v$ which reduces (2.11) to

$$\alpha = \beta', \tag{2.12a}$$

$$\alpha'' + \tfrac{1}{2}\alpha^2 + c\alpha - 1 = 0 \tag{2.12b}$$

($y = x - ct, \alpha = u$ and $\beta = v - t$). Again, (2.12b) is a direct reduction of (2.10) using G plus one integration. The solution of these is discussed in [9] but we look at a possible 'Lagrangian' picture in the next section.

We can make a similar analysis of (2.10) corresponding to the Galilean plus time invariant generator $X = t\partial/\partial x + a\partial/\partial t + \partial/\partial u$ (a is a nonzero constant). Here, of course, one will need to calculate the associated conservation law.

Recently [5], the above method has been successfully applied to the analysis of 'couette' flows that arise in fluid dynamics.

3. Lagrangians of reduced forms and double reductions

We now consider 'variational principles' for some p.d.e.s even though the p.d.e. itself does not admit a Lagrangian.

Illustrative example

In example 2, a reduction by G of the KdV equation yields the travelling wave solution by letting $y = x - ct$ and $w = u$, i.e.,

$$w''' + ww' = -cw'$$

which with one integration becomes

$$w'' + cw + \tfrac{1}{2}w^2 = 0 \tag{3.1}$$

(compare this with (2.12*b*) obtained using the symmetry/conservation law relationship). A discussion of the solution to (3.1) is given in [9]. It is interesting to note that even though the KdV equation is not derivable from a variational principle, a mathematical analysis of its reduced form, namely, (3.1) may appeal to a variational method as a Lagrangian of (3.1) is $L = \frac{1}{2}w'^2 - \frac{1}{2}cw^2 - \frac{1}{6}w^3$. Indeed, one has now the advantage of calculating Noether symmetries corresponding to L (and corresponding first integrals) to twice reduce (3.1).

Similarly, a reduction of the KdV equation for solutions invariant under X is dicussed in [9]. That is, if $y = x - \frac{1}{2}bt^2$ ($b = 1/a$) and $w = u - bt$, we get a third-order o.d.e which after one integration leads to a first Painlevé transcendent

$$w'' + \tfrac{1}{2}w^2 + by + k = 0. \tag{3.2}$$

A Lagrangian for (3.2) is $L^* = \frac{1}{2}w'^2 - \frac{1}{6}w^3 - \frac{1}{2}bw^2 + kw$. The second Painlevé transcendent corresponding to the KdV equation occurs by considering a scale-invariant reduction (see [9]).

This procedure has not been carried out previously and can be useful in the analysis of a large class of other nonlinear evolution-type equations such as the combined KdV-modified KdV equation

$$u_t + \alpha(1 + \beta u)uu_x + \gamma u_{xxx} = 0, \qquad \alpha, \gamma > 0, \tag{3.3}$$

which has recently attracted much attention. It can be shown that a similarity reduction leads to the second-order o.d.e.

$$\gamma w'' - \tfrac{1}{3}yw + \tfrac{1}{3}\alpha\beta w^3 = 0 \tag{3.4}$$

which has Lagrangian (for $\gamma = 1$) $L = \frac{1}{2}w'^2 + \frac{1}{6}yw^2 - \frac{1}{12}\alpha\beta w^4$. Surprisingly, (3.4) is also a second Painlevé transcendent.

A complete analysis of the Fisher equation in the manner described above is done below.

The Fisher equation

We consider, in detail, the Fisher equation

$$u_t = u_{xx} + \lambda u(1 - u) \tag{3.5}$$

which only admits point symmetries involving time and space translations. Also, (3.5) does not admit a Lagrangian.

A time translation reduction of the equation becomes the o.d.e. ($y = x$, $w = u$)

$$w'' + \lambda w(1 - w) = 0$$

and a translation in x reduction yields ($y = t$, $w = u$)

$$w' = \lambda w(1 - w).$$

A travelling wave reduction ($y = x - ct$ and $w = u$) from a combination of these Lie symmetry generators yields

$$w'' + cw' + \lambda w(1 - w) = 0. \qquad (3.6)$$

Equation (3.6) has a Painlevé property for $\lambda = \pm 6c^2/25$ (see [2]). We attempt an analysis and reduction of (3.5) using the notion of invariants in the way described above. Firstly, we note that a Lagrangian of (3.6) is

$$L = \mathrm{e}^{cy}\left[\tfrac{1}{2}w'^2 - \lambda\left(\tfrac{1}{2}w^2 - \tfrac{1}{3}w^3\right)\right]. \qquad (3.7)$$

The Noether symmetries $G = \xi\partial/\partial y + \eta\partial/\partial w$, if any, are given by solving

$$GL + L\frac{\mathrm{d}\xi}{\mathrm{d}y} = \frac{\mathrm{d}f}{\mathrm{d}y},$$

where $f = f(y, w)$ is some gauge term. Noether's theorem then provides a conserved quantity, $I = I(y, w, w')$, corresponding to each Noether symmetry G and gauge f. It is known that the first-order equation $I = k$ (k is a constant) which is a reduced form of (3.6) is also invariant under G which allows us to reduce once more with the symmetry G and, hence, to find a solution to (3.6) by quadrature being an exact solution of the Fisher equation. The calculations show that a Noether symmetry is $G = \mathrm{e}^{cy/5}\{\partial/\partial y + 2c/5(1 - w)\partial/\partial w\}$ ($f = (2/5)^2\,\mathrm{e}^{6cy/5}w(2 - w)$) coming from $\lambda = -6c^2/25$ which corresponds to the Painlevé property mentioned above. The corresponding first-order o.d.e. with G as Lie symmetry generator is

$$\mathrm{e}^{6cy/5}\left\{-\tfrac{1}{2}w'^2 - 2c^2/25w(1 - w)^2 + 2c/5w'(1 - w)\right\} = k. \qquad (3.8)$$

Equation (3.8) can be mapped to an equation in Y and W, $\bar{I}(W, W') = \bar{k}$ (a variables separable equation), i.e., with Lie symmetry $\bar{G} = \partial/\partial Y$ by solving the system of p.d.e.s $G(Y) = 1$ and $G(W) = 0$. We get $W = (1 - w)\,\mathrm{e}^{2cy/5}$ and $Y = (-5/c)\,\mathrm{e}^{-cy/5}$. After some lengthy calculations, (3.8) has the transformed form

$$\frac{\mathrm{d}W}{\sqrt{(2c/5)^2W^3 + 2k}} = -\mathrm{d}Y. \qquad (3.9)$$

The integral with respect to W is obtainable in terms of some special functions and substituting back for W, Y and w, y gives a nontrivial travelling wave solution for the Fisher equation. The better known analyses done on the Fisher equation or its reduced form are numerical (e.g., see [10, 11])—the above is completely analytical.

Note

Equation (3.6) generates the two-dimensional Lie algebra of point symmetries $G_1 = \partial/\partial y$ and G (above), the latter for $\lambda = -6c^2/25$ with the Lie bracket $[G, G_1] = (-c/5)G$. Thus, a solution by quadratures is obtainable, without recourse to Lagrangians, by reducing (3.6) first by G and then by G_1 (see [6]).

We now discuss a straight analysis of (3.5) by finding conserved vectors (T, S) on the general diffusion equation

$$u_t = u_{xx} + F(u). \qquad (3.10)$$

If such a vector exists, (3.10) and, in particular, the Fisher equation may be analysed with possible potential symmetries (see [3]) or we would be able to study the p.d.e. in the manner described in section 2. A conserved form of (2.9a) is

$$D_tT + D_xS = 0|_{(3.10)}. \qquad (3.11)$$

After expansion into partial derivatives, the tedious calculations reveal

$$T = ku_x + a(x,t)u + b(x,t), \qquad S = -ku_t - au_x a_x u + d(x,t) \tag{3.12}$$

subject to $b_t + d_x = 0$ and $a_t u + F(u)a + ua_{xx} = 0$. It is clear that (T, S) is a nontrivial conserved vector if $F = 0$ (heat equation) or $F = u$ (in which case we do not obtain the Fisher equation and a satisfies $a_t + a_{xx} + a = 0$). We conclude that the Fisher equation has no conservation laws (this implies that the analysis utilizing potential symmetries presented in section 2 cannot be done on the Fisher equation).

The Fitzhugh–Nagumo equation

We carry out a similar study of a generalized version of the Fitzhugh–Nagumo equation

$$u_t = u_{xx} + \lambda u(1-u)(u-a), \qquad a \neq 1. \tag{3.13}$$

The calculations, as before, are long and tedious but we briefly present the results. A Painlevé analysis has been done on (3.13) for $\lambda = 1$ in [4]. A travelling wave reduction $y = x - ct$ and $w = u$ yields the o.d.e.

$$w'' + cw' + \lambda w(1-w)(w-a). \tag{3.14}$$

A Lagrangian of (3.14) is $L = \mathrm{e}^{cy}\left[\frac{1}{2}w'^2 - \lambda\left(-\frac{1}{4}w^4 - \frac{a}{2}w^2 + \frac{a+1}{3}w^3\right)\right]$ which has a single Noether symmetry $G = \mathrm{e}^{cy/3}\left[\partial/\partial y + c/3\left(\frac{a+1}{3} - w\right)\partial/\partial w\right]$ for simultaneous forms of λ given by

$$\lambda = \frac{2c^2}{3(a^2 - a + 1)}, \qquad \lambda = \frac{4c^2}{9a} \tag{3.15}$$

from which we obtain $a = 1/2$ and $a = 2$ ($\lambda = 2\frac{4c^2}{9}$ and $\lambda = (1/2)\frac{4c^2}{9}$, respectively). Following from results regarding the Fisher equation, we propose that the equation (3.13) possesses Painlevé properties for these combinations of a and λ. Furthermore, the exact solutions for the equation are obtainable from a double reduction using the single point symmetry generator G.

The method enables one to analyse a large class of diffusion-type equations

$$u_t = u_{xx} + \lambda F(u), \tag{3.16}$$

which admit no Lagrangian but whose reduced form has a Lagrangian, in a variational way. We note that a travelling wave reduction always exists. Also, as a corollary, some new and useful Painlevé properties are obtainable.

4. Conclusion

We have presented the role of invariants in obtaining exact solutions of differential equations. Firstly, conserved vectors of a p.d.e. allow us to obtain reduced forms of the p.d.e. for which some of the Lie point symmetries are easily concluded. Secondly, invariants of reduced forms of a p.d.e. are obtainable from a variational principle even though the p.d.e. itself does not admit a Lagrangian. These reductions carry all the usual advantages regarding symmetries and further reductions as was seen in detail regarding the Fisher equation. The technicalities of 'symmetries of the manifold' and symmetries of a differential equation have only recently been a subject of study (see [8]). The analysis here emphasizes the existence of such a relationship implying that the Lagrangian/Hamiltonian structure in the reduced form of the evolution equation is not coincidental or merely a mathematical convenience but has an explanation in the structure of the underlying manifold or submanifold.

The method can be applied to a large class of evolution-type p.d.e.s, particularly diffusion-type equations. The method is a novel one and provides new solutions in some cases and proposes an alternative way of obtaining values of parameters for which certain classes of equations may possess Painlevé properties.

Acknowledgments

We thank the referees for making useful comments and suggestions which certainly lead to an improvement of the manuscript. Also, AHK thanks the NRF of South Africa for financial support under GUN 2054178.

References

[1] Abdel-Gawad H I 1995 On the invariants of nonlinear differential equations of third order *Nonlinear Dyn.* **8** 307–14

[2] Abdel-Gawad H I 2003 A method for finding the invariants and exact solutions of coupled non-linear differential equations with applications to dynamical systems *Int. J. Non-linear Mech.* **38** 429–40

[3] Bluman G and Kumei S 1986 *Symmetries and Differential Equations* (New York: Springer)

[4] Guo B Y and Chen X 1991 Analytic solutions of the Fisher equation *J. Phys. A: Math. Gen.* **24** 645–50

[5] Hayat T and Kara A H 2004 Couette flow of a third grade fluid with variable magnetic field *Math. Comput. Model.* (to appear)

[6] Ibragimov N H 1994 *CRC Handbook of Lie Group Analysis of Differential Equations* vol 1, ed N H Ibragimov (Boca Raton, FL: CRC Press)

[7] Kara A H and Mahomed F M 2000 The relationship between symmetries and conservation laws *Int. J. Theor. Phys.* **39** 23–40

[8] Khesin B 2005 Topological fluid dynamics *Not. AMS* **52** 9–19

[9] Olver P 1986 *Application of Lie Groups to Differential Equations* (New York: Springer)

[10] Parekh N and Puri S 1990 A new numerical scheme for the Fisher equation *J. Phys. A: Math. Gen.* **23** 1085–91

[11] Tang S and Weber R O 1990 Numerical study of Fisher's equation by a Petrov-Galerkin finite element method *J. Aust. Math. Soc. Ser.* B **33** 27–38